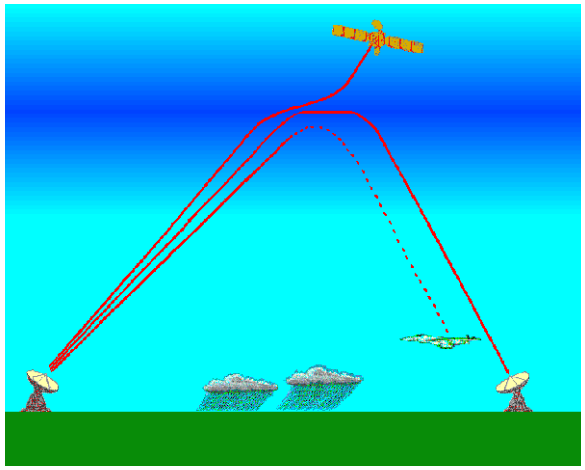
Dispersion and Group Velocity

In the simple lossless media we usually address in a basic course on electromagnetic theory, propagation constants are indeed constants. For example, in a dielectric medium (no magnetic properties) $k = \omega\sqrt{\mu_o \varepsilon} = \frac{\omega}{c}\sqrt{\varepsilon_r} = \frac{\omega}{c} n$ where c is the speed of light in vacuum and n is the index of refraction of the dielectric. Waves propagating in the $\pm z$ directions are represented with forms like $Ae^{\mp jkz + j\omega t}$. The phase velocity of such a wave is given by $u = \frac{1}{\sqrt{\mu_o \varepsilon}} = \frac{c}{n}$. The maximum achievable velocity is the speed of light in vacuum since $n \geq 1$. However, in a medium with dispersion, $k = k(\omega)$ or propagation properties depend on frequency. This property is called dispersion, because signals become dispersed. That is, since any signal carrying information must consist of more than one frequency, propagation that depends on frequency will distort the signal.

There are two simple examples of such frequency dependent propagation. The first is an ionized gas (plasma) such as is found in the earth's ionosphere. Here $k = k(\omega) = \frac{\omega}{c}\sqrt{1 - \frac{{\omega_p}^2}{\omega^2}}$ where $\omega_p = 2\pi f_p$ is the plasma frequency. This is a natural response frequency of the plasma and depends on the density of free electrons. From this expression, we see that $k = k(\omega) \approx \frac{\omega}{c}$ for frequencies well above the plasma frequency and $k = k(\omega) \approx \frac{\omega}{c}\sqrt{-\frac{{\omega_p}^2}{\omega^2}} \approx j\frac{\omega}{c}\sqrt{\frac{{\omega_p}^2}{\omega^2}} \approx j\frac{\omega_p}{c}$ for frequencies well below the plasma frequency. When k is real, waves propagate and when k is imaginary, they do not. Thus, for low frequencies, waves cannot propagate. This effect is readily seen in commercial radio. At AM frequencies (around 1MHz), signals reflect off of the ionosphere. This makes it possible to communicate around the curvature of the earth much further than at FM frequencies (around 100MHz) where no reflection occurs. Only line of sight communication is possible at FM frequencies. Thus, the plasma frequency must be somewhere between the AM and FM bands. It varies a great deal between day and night, but it is indeed generally in this range. A typical value is 30MHz. A picture from Chaisson and McMillan's book *Astronomy Today* shows some radio wave paths for satellite communication.



A second example of dispersive propagation is a standard parallel plate waveguide. Here $k = k(\omega) = \frac{\omega}{c}\sqrt{1-\frac{\omega_{co}{}^2}{\omega^2}}$, which has the same form as for a plasma except that $\omega_{co} = \frac{1}{\sqrt{\mu_o \varepsilon}}\frac{m\pi}{a}$ where *a* is the distance between the parallel plates and *m* is any positive integer. Again, we have assumed that the insulator between the plates is not magnetic. For very low frequencies, again there is no propagation. This effect is used in microwave ovens where small holes are cut into the cavity wall on the door of the oven so we can see what we are cooking without the microwave energy leaking out of the oven. Clearly high frequencies (like for visible light) still propagate through the holes since we can see through them. The wavelengths at microwave oven frequencies (*f=2.45GHz*) are about *10cm* so they are too large to fit in the holes. Actually, the holes must have a diameter of about a half wavelength for the waves to propagate. There is a second example of this cutoff effect that also involves AM and FM frequencies. If you listen to an FM station as you drive through a tunnel, you can usually detect the signal and hear the program throughout the tunnel, if at a somewhat reduced level. FM wavelengths are about *3m* so they can fit in the tunnel. AM wavelengths, on the other hand, are about *300m* so they do not fit in the tunnel and the waves cannot propagate. Even in a relatively short tunnel, AM signals will rapidly disappear. Try this experiment sometime.

The dependence of propagation on frequency produces some confusing results that require us to develop a more general picture of this process. The clearest problem occurs in the phase velocity $u = \frac{\omega}{k}$. For either a plasma or waveguide,

$u = \frac{\omega}{k} = \frac{\omega}{\frac{\omega}{c}\sqrt{1-\frac{\omega_{co}{}^2}{\omega^2}}} = \frac{c}{\sqrt{1-\frac{\omega_{co}{}^2}{\omega^2}}} \geq c$ for $\omega \geq \omega_p$ which is the condition required for propagation. Thus, the phase velocity exceeds the speed of light! We all know that it is not possible for EM waves to propagate faster than *c* so something must be wrong with this picture. The key to understanding this problem is that this velocity is the phase velocity. Such a velocity only characterizes the propagation of waves that exist for all time (steady-state). We are interested in waves that turn on and off and, thus, carry information. In communication systems there is usually some kind of a carrier frequency (as in AM and FM) with the signal of interest modulated on top of it. Signals come in clumps rather than continuously. Thus, we are really interested in how groups of signals propagate, not generally how single frequencies propagate. A more general velocity, called the group velocity, has been defined as $u_g = \frac{\partial \omega}{\partial k} = \frac{1}{\partial k / \partial \omega}$, which is equal to $u = \frac{\omega}{k}$ for nondispersive media. $u_g = \frac{1}{\partial k / \partial \omega} = \frac{1}{\partial / \partial \omega \left(\omega \sqrt{\mu_o \varepsilon}\right)} = \frac{1}{\sqrt{\mu_o \varepsilon}} = u$. This velocity does not exceed *c*. For plasmas or waveguides, we see that

$$u_g = \frac{1}{\partial k / \partial \omega} = \frac{1}{\partial / \partial \omega \left(\frac{\omega}{c} \sqrt{1 - \frac{\omega_{co}{}^2}{\omega^2}} \right)}.$$

$$\partial / \partial \omega \sqrt{\omega^2 - \omega_{co}{}^2} = \frac{1}{2} \frac{2\omega}{\sqrt{\omega^2 - \omega_{co}{}^2}} = \frac{\omega}{\sqrt{\omega^2 - \omega_{co}{}^2}}$$

$$u_g = \frac{1}{\partial k / \partial \omega} = \frac{c}{\omega} \sqrt{\omega^2 - \omega_{co}{}^2} = c \sqrt{1 - \frac{\omega_{co}{}^2}{\omega^2}} \le c$$

For lossless, dispersive media of this type, we see that

$$u u_g = c^2$$

which is a general result whose proof is addressed in advanced electromagnetic theory courses.

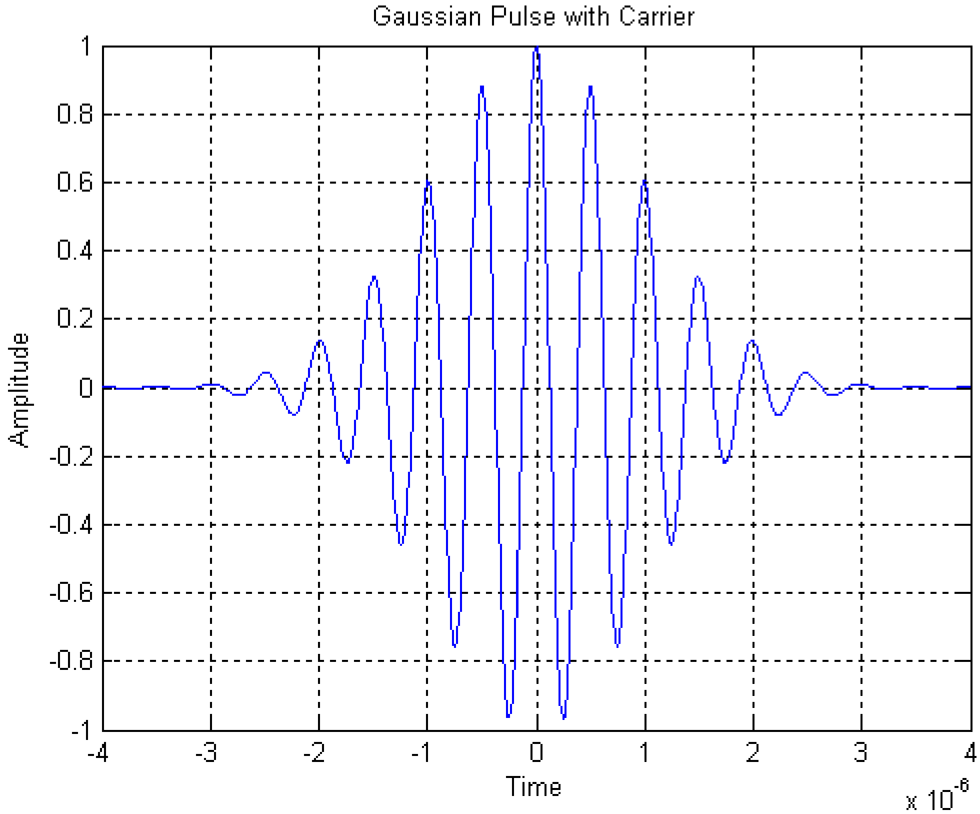
One of the most straight forward methods for investigating the form and meaning of the group velocity involves the use of the Fourier transform of a particular pulse shape – a Gaussian. Gaussian pulses have very useful mathematical properties and, more important to our purposes, they also nicely approximate pulses actually observed in practice. To obtain this transform, consider the following integral:

$$\int_{-\infty}^{+\infty} e^{-a^2 x^2} \cos(bx) dx = \frac{\sqrt{\pi}}{a} e^{-\frac{b^2}{4a^2}}$$

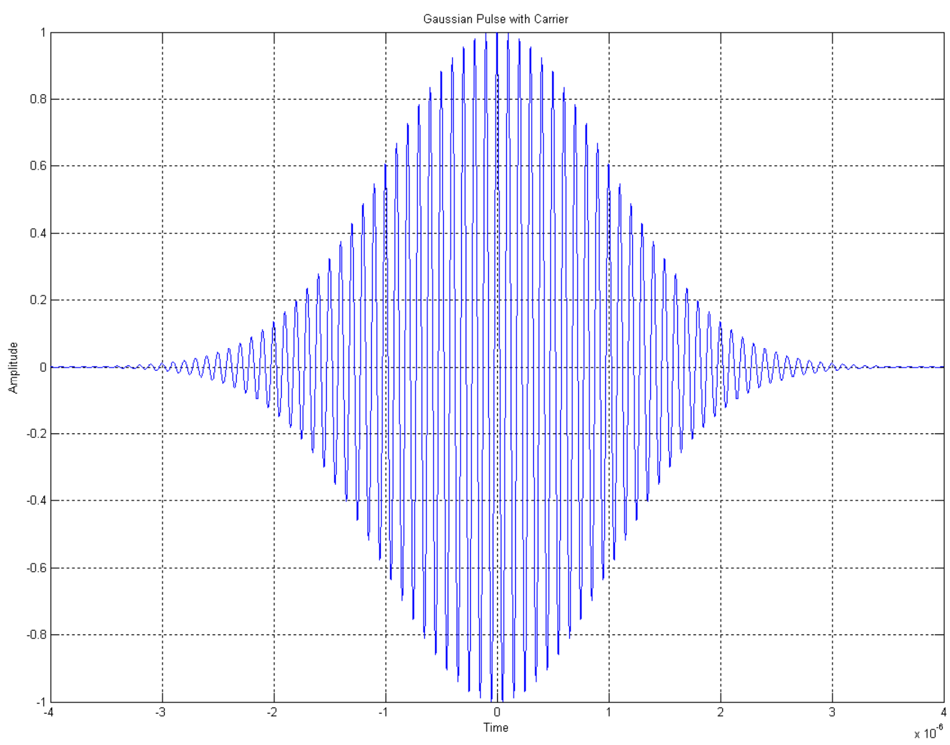
Then, if we assume an initial pulse at $z = 0$ of the form (this is a Gaussian pulse with a carrier frequency ω_o):

$$f(t) = e^{j\omega_o t} e^{-\frac{\sigma^2}{2} t^2}$$

For example, let $\omega_o = 2\pi f_o = 4\pi 10^6$ and $\sigma = 10^6$. Then this pulse has the form



For a much higher frequency $\omega_o = 2\pi f_o = \pi 10^7$, the envelope is more evident



The Fourier transform of this pulse will be:

$$F(\omega-\omega_o)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-j\omega t+j\omega_o t-\frac{\sigma^2}{2}t^2}\,dt=\frac{e^{-\frac{(\omega-\omega_o)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

Propagation away from $z = 0$ is accomplished by adding the spatial phase shift term to the integrand of the inverse transform:

$$f(z,t)=\int_{-\infty}^{+\infty} F(\omega-\omega_o)e^{-jk(\omega)z}e^{j\omega t}\,d\omega$$

To obtain this integral, we approximate the relationship between k and ω using the first three terms of the Taylor series expansion around ω_o

$$k(\omega)\approx k(\omega_o)+k'(\omega_o)(\omega-\omega_o)+k''(\omega_o)\frac{(\omega-\omega_o)^2}{2}+\ldots$$

Grouping similar terms allows us to recognize the general form of the integral

$$f(z,t)\approx e^{-jk(\omega_o)z+j\omega_o t}\int_{-\infty}^{+\infty} F(\omega-\omega_o)e^{j(\omega-\omega_o)[-k'(\omega_o)z+t]}e^{-jk''(\omega_o)\frac{(\omega-\omega_o)^2}{2}z}\,d\omega$$

Note that the last term involving k'' can be combined with $F(\omega-\omega_o)$ and the term $[-k'(\omega)z+t]$ is a retarded time. Thus, the final form of the integral for $f(z,t)$ is

$$f(z,t)\approx \frac{e^{-jk(\omega_o)z+j\omega_o t}}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} e^{j\Omega[-k'(\omega_o)z+t]}e^{-\frac{\Omega^2}{2\sigma^2}[1+jk''(\omega_o)\sigma^2 z]}\,d\Omega$$

which becomes, finally

$$f(z,t)\approx e^{-jk(\omega_o)z+j\omega_o t}\,\frac{e^{-\frac{\sigma^2}{2}[t-k'(\omega_o)z]^2}}{\sqrt{1-j\sigma^2 k''(\omega_o)z}}$$

We can therefore see that the Gaussian pulse propagates at the group velocity $1/k'(\omega_o)$ and spreads out as z increases. All the effects of dispersion are evident in this expression since the amplitude decreases as z increases. Since most physical pulses look something like a Gaussian, this result is reasonably general. In fact, by invoking the concepts of geometrical optics, the results can be shown to be quite general. Note also that, for non-dispersive media (like simple dielectrics), $k''=0$ and $k'=k/\omega$ and pulses propagate undistorted at the phase velocity.